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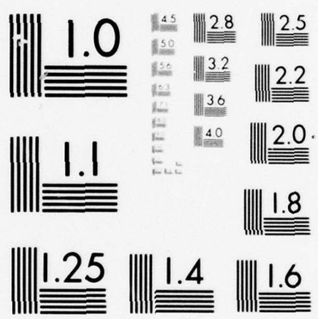


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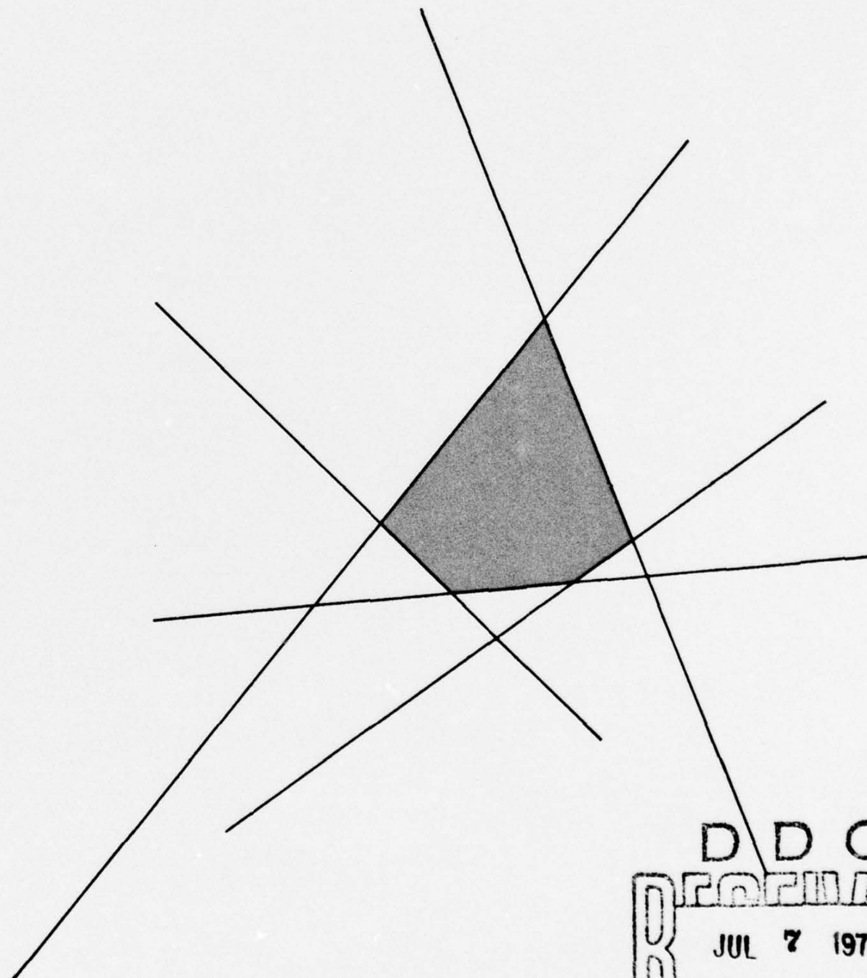
# OPTIMAL INSPECTION POLICIES FOR DETERIORATING MARKOV PROCESSES

by
ROBERT D. LEVIN





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OPTIMAL INSPECTION POLICIES FOR DETERIORATING MARKOV PROCESSES

Operations Research Center Research Report No. 77-8

Robert D. Levin

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## ABSTRACT

A model is presented of a Markov process whose state is unknown except when an inspection is performed. The evolution of the process is governed by fixed transition probability matrices $P$ and $Q$ under non-inspection and inspection, respectively. The costs of inspection and non-inspection depend on the current state. The objective is to characterize inspection policies which minimize expected total discounted cost.

The following specific models are presented. Model I is a process which starts in state $0$ and is terminated when, on inspection, the state is found to exceed some fixed value $M$. In Model II the process is repaired (reverts to state $0$) when the state at inspection exceeds $M$. Simple conditions are given which imply that the optimal inspection interval is a non-increasing function of the last observed state.

Model III is an inventory process with uncertain supply as well as demand. Given order size $n$, the number received is Binomial $(n;p)$. Costs of ordering, storage, and shortage are incorporated. In the single period case, conditions are given which imply that the optimal order size is non-increasing in current inventory. This result extends to the undiscounted multiperiod case provided the holding cost is zero. A counterexample is given for a two period case with linear, non-zero holding, shortage, and ordering costs.

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# CHAPTER 1

# MARKOV INSPECTION PROCESSES: INTRODUCTION AND APPLICATIONS

## 1.1 Introduction

We will be concerned with processes for which the state is not known with certainty unless a specific act of inspection is performed. Once the state is determined, a maintenance action is chosen according to a policy which is fixed in advance. Whether or not the process is inspected, it undergoes a random transition into a new state. Thus, at each time period, a choice must be made: to operate the process without inspection, or to inspect, and possibly maintain the process.

A variety of problems which have appeared in the literature fit this model. The applications we will consider fall under three main headings: terminating processes, repairable processes, and an inventory model with uncertain production. In this chapter we will discuss briefly each of these models, and give applications. In Chapter 2, a common mathematical framework for these models is developed. We prove a general theorem about the structure of optimal policies. In Chapters 3 and 4 we explore in detail the models presented in this chapter, and apply the results of Chapter 2.

## 1.2 A Brief Summary of the General Model

Our underlying process is discrete in time, and its state can be described by an integer. In general the state is not known.

If the process is not inspected, a state dependent cost is incurred. The process undergoes a random transition according to a transition probability matrix $P$ . The state remains unknown.

If the process is inspected, again a state dependent cost is incurred. The process undergoes a transition according to a (possibly different) probability matrix $Q$ . Thus, the act of inspection *may* affect the underlying process. In addition, under inspection the next state of the process becomes known.

We shall assume throughout that the process is Markovian. This means, given the present state and decision, that future states (and costs) are conditionally independent of past states and decisions.

The process described above will be called a Markov inspection process. For a more formal definition, see Chapter 2.

We will be concerned with characterizing the structure of policies which minimize the expected total discounted cost (or average cost, where appropriate) of operating the process.

## 1.3 Model I: Terminating Inspection Processes

In this model we will assume that the process has a finite state space $\{0,1, \ldots, N\}$ . When inspection occurs and the process is discovered to be in a state $j$ which exceeds some critical value $M$ , the process terminates.

The terminating process model may be applied to the problem of minimizing the operating cost of a non-repairable machine. We suppose the machine can be in any one of $N + 1$ states ranging from $0$ (perfect) to $N$ (failed). Transitions occur according to a Markov chain and the state remains unknown unless the machine is inspected. The cost of inspection depends on the state, and there is a cost per period for undetected failure.

In Chapter 3 we give conditions such that the optimal number of periods between successive inspection is a nonincreasing function of the state at last inspection.

A second example of a terminating inspection process is the "optimal reject allowance" problem. In its simplest formulation, a shop must fill an order for $N$ items. It costs $C_o + kC$ to produce a lot of $k$ items. Each item has a fixed probability $q$ of being defective, independently of the others in the lot. Thus the number of usable items resulting from a lot of size $k$ has distribution Binomial $(k;1 - q)$. The problem is to compute the optimal lot size $k^*(N)$, or equivalently, the optimal reject allowance $k^*(N) - N$. This problem is studied in detail by A. Beja [ 1 ].

We may consider Beja's model as a terminating inspection process as follows. The state is the number of good items on hand, or $N$, whichever is smaller. If $k$ items of the present lot have been produced but not inspected, we may produce another item without inspection, or terminate production and determine the number of good items. The latter action corresponds to inspection. If the state, after inspection, is less than $N$, a new lot must be started and a set-up cost $C_o$ is incurred. (This corresponds to an inspection cost in our general model of Chapter 2). If the state, after inspection, is $N$, the process terminates.

Beja proves that the optimal lot size $k^*(N)$ is nondecreasing in $N$. He also shows that, if $f(k)$ is the cost of first producing a lot of size $k$, and then continuing with an optimal policy, then $f(k)$ is quasi-convex in $k$. That is,

$$f(k+1) \leq f(k) \quad \text{for} \quad k < k^*(N)$$

and

$$f(k+1) \geq k(k) \quad \text{for} \quad k \geq k^*(N) \ .$$

These results, besides being of theoretical interest, have computational implications which are useful in finding optimal policies.

In Chapter 3, using our Markov inspection process formulation, we will extend these results to the case where the output from a lot of size $k$ is determined by $k$ transitions of a Markov chain. This includes the binomial case, the batch production case, and, in particular, the case where the number of usable items has the Poisson distribution with parameter $k\lambda$ .

## 1.4 Model II: Inspection Processes with Repair

This model differs from the terminating inspection model in that, when an inspection takes place and the state is discovered to exceed a critical value $M$ , the process is repaired and reverts to state $0$ instead of terminating.

A motivating example is the production process described by Sackrowitz and Samuel-Cahn [11], in which the states of successive units produced form a Markov chain. A unit may be inspected at fixed cost, and, if defective, may be replaced by a perfect item. If not inspected, the item becomes part of the output process without being observed. Under the assumption that the Markov chain is irreducible, the authors prove that a stationary inspection policy is optimal. That is, if the last item inspected was in state $i$ , it is optimal to produce a fixed number $t_i$ of items before inspecting again.

In the present work we are assuming that the process cannot improve (decrease in state) spontaneously. We feel this is more realistic than the assumption of irreducibility. When state N is reached, and discovered via an inspection, the process is repaired and reverts to state 0 . In Chapter 3 we will present conditions under which the optimal inspection intervals $t_i$ are nonincreasing in i .

Much work has been done on the related three action model where no action, inspection, and repair are independent alternatives. In particular, S. Ross [10] considered the observed state space to be $\{P = (P_0, P_1, \ldots) : P_i \geq 0 , \sum P_i = 1\}$ with the interpretation that $P_i$ is the posterior probability that the underlying process is in state i . He proves that the optimal inspect and repair regions are convex. For the two-state problem this implies a four region structure for the optimal policy: Take no action if $0 \leq p < p_1$ , inspect if $p_1 \leq p < p_2$ , take no action if $p_2 \leq p < p_3$ , and repair if $p_3 \leq p \leq 1$ . (Here p is the probability of being in the failed state).

In our model there will be only one critical number for the two-state case: if $p < p_1$ take no action; if $p \geq p_1$ inpsect (and repair if necessary).

Ross' work was generalized by D. Rosenfield [ 8 ], who presents a model in which the optimal policy can be characterized by three critical numbers for each underlying state. Under specified conditions on the costs and transitions matrix, Rosenfield proves that, if the last known state was $i$ and $k$ transitions have since occurred, it is optimal to take no action if $0 \leq k < k_1(i)$, inspect if $k_1(i) \leq k < k_2(i)$, take no action if $k_2(i) \leq k < k_3(i)$, and repair if $k \geq k_3(i)$. Two or more of the critical numbers may coincide for a given $i$. In addition, he shows that $k_1(i)$ and $k_3(i)$ are nonincreasing in $i$.

Under conditions similar to those given by Rosenfield, we will show in Chapter 3 that an analogous result holds for our two action model. That is, the optimal policy is specified by a single non-increasing sequence of critical numbers $k(i)$. It is optimal to take no action if $k < k(i)$ and inspect (and repair if necessary) if $k \geq k(i)$.

We feel that the present model has several advantages over Rosenfield's. In our model repair must be preceded by inspection. This is realistic in many physical situations. Consider, for example, a complex system of components operating in a "black box." Suppose that, once the box is opened, the state of the system becomes known. If it is necessary to open the box in order to repair the system, then the present model, rather than Rosenfield's, is applicable.

Furthermore, our results are valid in somewhat greater generality. Rosenfield assumes that the cost of inspection is a constant surcharge over the cost of non-inspection. We need only a weaker assumption that the cost of non-inspection is nondecreasing relative to the cost of inspection as the state increases.

Finally, since our inspection repair process is a two action problem rather than a three action problem, optimal policies are much easier to compute and administer.

## 1.5 Model III: An Inventory Process with Uncertain Production

In this model random demands for a commodity occur over a number of periods. The problem is to minimize the expected discounted sum of production, storage, and shortage costs. As stated thus far, this is the usual dynamic inventory model (see for example [ 5 ]).

The distinctive feature of this model is that the output from a given production lot or order is random. The simplest case is the binomial case where, when $n$ items are produced the output of usable items is distributed Binomial $(n,p)$ for some fixed $p$. More generally, the model includes the case of random batch production and, in particular, the case of Poisson production.

We assume a linear cost of production (no set-up cost). A storage cost $h(t)$ is imposed in each period for $t$ units of inventory in excess of demand, and a shortage cost $k(t)$ is imposed for $t$ units short of demand. Unsatisfied demand is carried over (backlogged) to the next period.

A similar model was considered, for the one period case, by Karlin [ 4 ]. Under the assumptions that the holding and shortage costs are convex, he proves that the optimal production level $n^*(x)$ is nonincreasing in current inventory $x$. (Intuitively - the less stock on hand, the more should be produced).

However, Karlin's approach does not extend to the multiperiod case. The reason is that the $n$ period optimal cost function need not be convex. A counterexample is given in Chapter 4.

In Chapter 4 we reprove and extend Karlin's result, without assuming convexity but with some restriction on the distribution of demand (assumed to be $PF_2$ , see [ 6 ]).

We proceed by formulating the problem as a Markov inspection process whose state is the current inventory. The production decision is regarded as being made sequentially. Thus, suppose $t$ items (or batches) have been produced in the current period, but the output of usable items has not been determined. Either an additional item (batch) may be produced, or production for the period may be terminated and demand satisfied. The latter action corresponds to inspection, in our general model, and the former, to non-inspection.

Applying the theory developed in Chapter 2, we prove that Karlin's intuitive result extends to the multiperiod case, provided there is no storage cost and no discounting.

When there is a storage cost or discounting the result need not hold. In Chapter 4 we present an example of a two period problem with constant demand and linear production, storage, and shortage costs for which monotonicity of the optimal production level is violated. In this example it is optimal to produce one item when initial inventory is three items, but optimal production is zero when initial stock is two items. Thus, our intuition may fail even in a very simple multiperiod problem.

In the case where there is no storage cost, we are able to obtain the result that, if it is optimal to produce when inventory is $x$ and $L$ periods remain, then it is optimal to produce when inventory is $x$ and $L'$ periods remain, $L' > L$ . We are unable to obtain the stronger result that the optimal production level is nondecreasing in the number of periods remaining. This seems to be an open question.

For the one period case with linear holding and shortage costs and Binomial $(n;p)$ production, we show that the optimal lot size for a given initial inventory is nonincreasing in the unit ordering and holding costs and nondecreasing in the unit shortage cost. The optimal cost is nonincreasing as a function of $p$.

# CHAPTER 2

# MARKOV INSPECTION PROCESSES: THE GENERAL MODEL

## 2.1 Description of the Model

Assume that the state of a process at time $n$ is given by an integer $X_n$ . In general, we will assume that $X_n$ is not known with certainty. What is known about the process can be expressed by the ordered pair $Y_n = (i,t)$ , where $i = X_{n-t}$ is the most recent state known with certainty and $t$ periods have elapsed since the state was last determined.

In any observed state $(i,t)$ there are two possible actions:

action 0: non-inspection

action I: inspection.

Let $A_n$ denote the action taken at time $n$ .

Under action 0 the process undergoes a transition according to the transition probability matrix $P$ . Thus $P(X_{n+1} = k \mid X_n = j , A_n = 0) = P_{jk}$ independent of the history of the process. In addition, if $X_{n+1} = k$ , a cost $0_k \geq 0$ is incurred and no new information about the process is obtained. Thus if $Y_n = (i,t)$ and $A_n = 0$ , the process undergoes $t + 1$ consecutive transitions according to $P$ between $X_{n-t} = i$ and $X_{n+1} = j$ , so that $P(X_{n+1} = j \mid Y_n = (i,t) , A_n = 0) = P_{ij}^{t+1}$ . The expected cost incurred will be $C(Y_n,A_n) = C(i,t;0) = \sum P_{ij}^{t+1} 0_j$ . Since no information is gained, the new observed state will be $Y_{n+1} = (i,t + 1)$ .

Under action I the process undergoes a transition according to the transition probability matrix $Q$. Thus $P(X_{n+1} = k \mid X_n = j, A_n = I) = Q_{jk}$, and again $X_{n+1}$ is conditionally independent of the history of the process, given $X_n$ and $A_n$. If $X_{n+1} = k$, a cost $I_k \geq 0$ is incurred. Moreover, the new state $X_{n+1}$ becomes known. Thus we interpret action I as inspection. If $Y_n = (i,t)$ and $A_n = I$, the process undergoes $t$ consecutive transitions under $P$, followed by a transition under $Q$, between states $X_{n-t} = i$ and $X_{n+1} = j$. Thus $P(X_{n+1} = j \mid Y_n = (i,t)$ and $A_n = I) = \sum P^t_{ik}Q_{kj}$. It follows that under action I the expected cost incurred will be $C(Y_n,A_n) = C(i,t;I) = \sum_{j,k} P^t_{ik}Q_{kj}I_j$; and the next observed state will be $Y_{n+1} = (j,0)$ with probability $\sum_k P^t_{ik}Q_{kj}$.

Definition:

We will call such a process a Markov inspection process.

As will be seen in subsequent chapters, the models presented in Chapter 1 are all examples of Markov inspection processes.

## 2.2 Definition and Existence of Discounted Cost Optimal Policies

We will study the problem of minimizing the expected discounted total cost of the Markov inspection process described in 2.1.

We will assume that there is no upper bound to the number of non-inspections (actions 0) between two successive inspections (actions I), but the process terminates after $L(\leq \infty)$ inspections take place.

In some applications, e.g. models I and II of Chapter 1, a fixed period of time elapses between transitions. In this case future costs are discounted at a rate $\alpha \leq 1$ per transition.

In other applications, e.g. our inventory model (model III of Chapter 1), transitions involving non-inspection are instantaneous and one time unit passes between successive inspections. In this case future costs are discounted at a rate $\beta \leq 1$ per inspection.

In the remainder of this section we will treat both cases simultaneously. On our applications, however, either $\alpha = 1$ or $\beta = 1$ . Note that we are also considering the undiscounted case $\alpha = \beta = 1$ .

Let $\pi$ be a policy, i.e. any rule for choosing actions. Let $Y_n(\pi)$ be the observed state at time $n$ , under policy $\pi$ . Let $A_n(\pi)$ be the action chosen by $\pi$ at time $n$ . Let $T_\ell(\pi)$ be the number of transitions under $\pi$ up to the $\ell$-th inspection. Let $T_o(\pi) = 0$ and fix $\alpha$ , $\beta \in (0,1]$ .

Define the expected discounted cost of starting in $Y_o = (i,t)$ and operating under $\pi$ to be

(2.2.1)
$$V(i,t,L,\pi) = E_\pi\left[\sum_{\ell=0}^{L-1} \beta^\ell \sum_{n=T_\ell(\pi)+1}^{T_{\ell+1}(\pi)} \alpha^n C(Y_n(\pi),A_n(\pi)) \mid Y_o = (i,t)\right]$$

where $E_\pi$ denotes the expected value with respect to the distributions of $Y_n(\pi)$ , $A_n(\pi)$ , and $T_\ell(\pi)$ induced by $\pi$ . Note all costs are nonnegative so that $E_\pi$ is well defined.

Let $V^*(i,t,L) = \inf_\pi V(i,t,L,\pi)$ .

Definition:

A policy $\pi^*$ is optimal if

$$V(i,t,L,\pi^*) = V^*(i,t,L) \quad \text{for all} \quad i\ ,\ t \quad \text{and} \quad L\ .$$

We are now ready to state a theorem on the existence of optimal policies.

Theorem 2.2:

There exists an optimal policy $\pi$ such that $A_n(\pi)$ is a deterministic function of the current observed state $Y_n(\pi)$ and the number of inspection opportunities remaining, $L$ . That is, $A_n = f(Y_n(\pi),L)$ .

Furthermore $V^*(i,t,L)$ satisfies the following optimality equation:

$$V^*(i,t,L) = \min\left\{\sum P_{ij}^{t+1} O_j + \alpha V^*(i,t+1,L)\ ; \quad \sum P_{ij}^{t} Q_{jk}[I_k + \alpha\beta V^*(k,0,L-1)]\right\} \tag{2.2.2}$$

and $f(i,t,L)$ may be any function such that $f(i,t,L) = 0$ when the first term in (2.2.2) is minimal and $f(i,t,L) = I$ when the second term is minimal.

If $\alpha < 1$ , and the sequences $\{O_j\}$ , $\{I_j\}$ are bounded, then (2.2.2) determines $V^*(i,t,L)$ uniquely.

Proof:

The last result is a standard theorem in discounted dynamic programming [ 2 ]. The remainder of the theorem, which holds even when $L = \infty$ , and $\alpha = \beta = 1$ , follows, since all costs are non-negative, from standard results in negative dynamic programming [12].■

Remark:

The first term in the optimality equation (2.2.2) is the expected cost of the policy: perform action 0, then proceed optimality. The second term is the expected cost of the policy (which we will call $\pi_0$): perform action I, then proceed optimality. Thus the optimal action is that action which yields the minimum cost, when all subsequent actions are optimal. This is just the well known principle of optimality.

## 2.3 A Structure Theorem

In this section we will prove a general result which will apply to the models presented in Chapter 1.

We begin with some definitions.

Definition:

A policy $\pi$ is said to be monotone if, for each $L$ , there exists a nonincreasing sequence $t^*(i,L)$ such that

$$A_n(\pi) = \begin{cases} I & \text{if } Y_n = (i,t) \text{ and } t \geq t^*(i,L) \\ 0 & \text{if } Y_n = (i,t) \text{ and } t < t^*(i,L) \end{cases}$$

when $L$ inspection opportunities remain.

Let $\pi_0$ be the policy: perform action I, then proceed optimally.

Let $\pi_1$ be the policy: perform action 0, then proceed according to $\pi_0$ .

Definition:

A policy $\pi$ is said to be a one stage look-ahead policy if $A_n(\pi) = I$ if and only if $V(Y_n(\pi);L,\pi_0) < V(Y_n(\pi);L,\pi_1)$ i.e. $\pi$ inspects whenever immediate inspection is cheaper than non-inspection followed by inspection, subsequent actions being optimal.

Remark:

The terminology "one stage look-ahead policy" comes from the theory of optimal stopping problems [ 9 ]. In the present case we identify action I with "stopping" and paying a terminal cost equal to the expected cost of acting optimally for the remainder of the process. Action 0 is identified with "continuing." A one stage look-ahead policy stops exactly when stopping is less costly than continuing one more stage and then stopping.

Conditions for a one stage look-ahead policy to be optimal can be found in [ 9 ].

The purpose of this section is to give sufficient conditions for the existence of an optimal policy that is a monotone, one stage look-ahead policy.

We will need to make the following assumptions on the transition matrix P :

Assumptions:

A1: $P$ is upper triangular, i.e. $P_{ij} = 0$ for $j < i$ ; and $P_{ii} < 1$ . (If the number of real states is finite $(N + 1)$ we allow $P_{NN} = 1$) .

A2: $P$ is $TP_2$ (totally positive of order 2), i.e. $P_{ij}P_{i'j'} - P_{ij'}P_{i'j} \geq 0$ for all $i \leq i'$ , $j \leq j'$ .

Assumption A1 means that the real state of the process cannot decrease under action 0, and, with positive probability, may increase. Assumption A2 is a technical condition which implies among other things that $P$ satisfies the increasing failure rate (IFR) property [ 3 ], i.e. $P(X_{n+1} \geq j \mid X_n)$ is an increasing function of $X_n$ for each $j$ . The $TP_2$ property is needed in the proof of Lemma 2.3.1 to follow. A complete discussion of total positivity is given in [ 6 ].

We will also require the following

Structural Conditions:

(SC1): There exist critical numbers $j_L$ such that $V(j,0,L,\pi_0) \geq V(j,0,L,\pi_1)$ for $j < j_L$ and $V(j,0,L,\pi_0) < V(j,0,L,\pi_1)$ for $j \geq j_L$ . (Thus if the true state is known to be $j \geq j_L$ , then immediate inspection is less costly than waiting one period and inspecting).

(SC2): For each $(i,t)$ there exists $t' \geq t$ such that inspection is strictly optimal in state $(i,t')$ , when $L$ inspection opportunities remain.

The above structural conditions will be verified for each of the models presented in Chapter 1, in succeeding chapters.

We are now ready to state the main theorem of this chapter.

Theorem 2.3:

If assumptions A1 and A2 are satisfied, and the structural conditions (SC1) and (SC2) hold, then a monotone, one stage look-ahead policy is optimal.

Before beginning the proof, we will need one more definition and a lemma.

Definition:

A sequence $\Delta_j$ is said to have the single crossing property (SCP) if $\Delta_j > 0$ implies $\Delta_k > 0$ for all $k > j$ .

Remark:

The usual definition is somewhat weaker: $\Delta_j > 0 \Rightarrow \Delta_k \geq 0$ for $k > j$ . This is not sufficient, without further argument, to ensure monotonicity of the critical numbers. See the appendix to this chapter for further discussion and a proof of the following lemma.

Lemma 2.3:

If $\Delta_j$ satisfies the SCP in $j$ and $P$ satisfies A1 and A2, then $\sum P_{ij}^t \Delta_j$ satisfies the SCP in $i$ and $t$ .

Proof:

See the appendix to this chapter. ■

Proof of Theorem 2.3:

Defining $\Delta(i,t,L) = V(i,t,L,\pi_1) - V(i,t,L,\pi_0)$ and conditioning on the true state $X_n$ given $Y_n = (i,t)$ yields:

$$(2.3.1) \qquad \Delta(i,t,L) = \sum P^t_{ij} \Delta(j,0,L) .$$

(This key result may also be seen algebraically by direct expansion of $V(i,t,L,\pi_0)$ and $V(i,t,L,\pi_1)$) .

The first structural condition (SC1) is equivalent to $\Delta(j,0,L) \leq 0$ , $j < j_L$ and $\Delta(j,0,L) > 0$ for $j \geq j_L$ , i.e. $\Delta(j,0,L)$ has the SCP in $j$ .

It immediately follows from Lemma 2.3 and (2.3.1) that $\Delta(i,t,L)$ has the SCP in $i$ and $t$ .

We can now define the critical numbers $t^*(i,L) = \min\{t : \Delta(i,t,L) > 0\}$ .

By (SC2), for some $t_i$ , inspection is strictly optimal in $(i,t_i)$ , which implies that $\Delta(i,t_i,L) > 0$ . Hence $t^*(i,L) < \infty$ .

By the SCP in $t$ , $\Delta(i,t,L) \leq 0$ for $t < t^*(i,L)$ and $\Delta(i,t,L) > 0$ for $t \geq t^*(i,L)$ .

Furthermore, since $\Delta(i,t,L)$ satisfies the SCP in $i$ , we must conclude that $t^*(i,L)$ is nonincreasing in $i$ . (Otherwise, if $t^*(i+1,L) > t^*(i,L)$ , then, for some $t$ , $\Delta(i,t,L) > 0$ and $\Delta(i+1,t,L) \leq 0$ , contradicting the SCP).

Let us define the policy $\pi$ by $A_n(\pi) = I$ if and only if $L$ inspection opportunities remain, $Y_n = (i,t)$ , and $t \geq t^*(i,L)$ . Then $\pi^*$ is a monotone policy. $\pi^*$ is also a one stage look-ahead policy, since $\pi^*$ inspects in state $(i,t)$ if and only if immediate inspection is less costly than non-inspection followed by inspection, subsequent actions being optimal.

It remains only to prove that $\pi^*$ is optimal.

Let $t < t^*(i,L)$ . Then $V(i,t,L,\pi_1) \leq V(i,t,L,\pi_0)$ .

The term in the optimality equation (2.2.2) corresponding to action 0 is

$$\sum P_{ij}^{t+1} O_j + \alpha V^*(i,t+1,L) \leq \sum P_{ij}^{t+1} O_j + \alpha V(i,t+1,L,\pi_0) =$$

$$V(i,t,L,\pi_1) \leq V(i,t,L,\pi_0) ,$$

which is the term of (2.2.2) corresponding to inspection. It follows from Theorem 2.2, that action 0 is optimal in state $(i,t)$ when $t < t^*(i,L)$ .

We next consider the case $t \geq t^*(i,L)$ . Suppose, to obtain a contradiction, it is strictly optimal to perform action 0 when $Y_n = (i,t')$ , $t' \geq t^*(i,L)$ . Let $t'' = \min\{t > t' : \text{action I is optimal in } (i,t)\}$ . (The above set is non-empty by (SC2)). Note that action 0 is optimal in $(i,t''-1)$ and action I is optimal in $(i,t'')$ . This implies that $\pi_1$ is an optimal policy in state $(i,t''-1)$ , hence $V(i,t''-1,L,\pi_1) \leq V(i,t''-1,L,\pi_0)$ . But $t''-1 \geq t^*(i,L)$ so $\Delta(i,t''-1,L) > 0$ , yielding a contradiction. It follows that action I is optimal for all $t \geq t^*(i,L)$ .

Thus we have shown that $\pi^*$ is optimal, and the proof is complete.■

Appendix to Chapter 2:

One of the major properties of $TP_2$ matrices is that they preserve the single crossing property. In this appendix we will use the term weak single crossing property (WSCP) for the usual definition, with weak inequality instead of strict inequality. (See the remark following the definition of the single crossing property).

Karlin [ 6 ] proves that if $\Delta_j$ has the WSCP in $j$, then $\sum P_{ij}\Delta_j$ has the WSCP in $i$. In [ 7 ] Rosenfield shows that if in addition to being $TP_2$ (A1), $P$ is upper triangular (A2) then $\sum P^t_{ij}\Delta_j$ has WSCP in $i$ and $t$.

We wish to extend these results to the strong version (SCP) defined above.

Let $\Delta(j)$ have the SCP and $P$ satisfy A1 and A2. Let $\Delta(i,t) = \sum P^t_{ij}\Delta(j)$. Suppose $\Delta(i_0,t_0) > 0$, and $t_1 > t_0$. By Rosenfield's lemma, $\Delta(i_0,t_1) \geq 0$. Suppose, to obtain a contradiction, $\Delta(i_0,t_1) = 0$. Let $0 < \varepsilon < \Delta(i_0,t_0)$ and let

$$\Delta'(j) = \begin{cases} \Delta(j) - \varepsilon & \text{if } \Delta(j) \leq 0 \\ \Delta(j) & \text{if } \Delta(j) > 0 \end{cases}.$$

Then $\Delta'(j)$ satisfies the SCP. Letting $\Delta'(i,t) = \sum P^t_{ij}\Delta'(j)$, we have $\Delta(i,t) - \varepsilon \leq \Delta'(i,t) \leq \Delta(i,t)$. Hence $\Delta'(i_0,t_0) > 0$. Now $\Delta'(i,t)$ satisfies the WSCP in $t$ so that $\Delta'(i_0,t_1) \geq 0$. But $\Delta'(i_0,t_1) < \Delta(i_0,t_1) = 0$, yielding a contradiction. Thus $\Delta(i,t)$ has the SCP in $t$. The same proof shows that $\Delta(i,t)$ has the SCP in $i$. This completes the proof of Lemma 2.3. ■

# CHAPTER 3

# MODELS OF INSPECTION FOR TERMINATING AND REPAIRABLE PROCESSES

## 3.1 INTRODUCTION

In this chapter we will characterize the structure of optimal policies for the terminating and repairable processes (models I and II) introduced in Sections 1.3 and 1.4

We will assume that the underlying process has a finite state space $\{0,1, \ldots, N\}$ . When inspection takes place and the state is observed, termination or repair occurs according to a policy which is fixed in advance. This policy will be assumed to have the following form: Terminate (repair) the process when the observed state $j$ exceeds some critical value $M \leq N$ . Otherwise the process is not altered by inspection.

The repairable model fits our general scheme of Chapter 2 as follows: Under non-inspection the transition probability is $P(X_{n+1} = j \mid X_n = i, A_n = 0) = P_{ij}$ as in Chapter 2. Under inspection, the transition probability matrix $Q$ is related to $P$ by

$$Q_{ij} = P(X_{i+1} = j \mid X_n = i, A_n = I) = \begin{cases} P_{io} + \sum_{j=M}^{N} P_{ij} & j = 0 \\ P_{ij} & 1 \leq j < M \\ 0 & j \geq M . \end{cases}$$

Thus, under inspection the process is returned to state $0$ if it would otherwise have been in a repairable state $(\geq M)$ after a transition according to $P$ . For $j \geq M$ the inspection cost $I_j$ is considered to include the cost of repair.

Remark:

Under our assumptions on the costs and transition matrix, if the underlying state $j$ were known at all times, the optimal repair policy would have the structure given above. That is, it is optimal to repair exactly when $j \geq M$ for some $M$. More general sufficient conditions for this type of policy, called a "control limit policy" to be optimal, are given by Derman [3].

To show that the terminating process also fits our general model, we will define a terminated state $N^*$. The process enters state $N^*$ only under inspection when the state would otherwise have been $\geq M$ under a transition according to $P$. Thus the transition matrix $P$ is defined as in Chapter 2 by $P_{ij} = P(X_{n+1} = j \mid X_n = i, A_n = 0)$ with the additional conditions that $P_{iN^*} = 0$, $i \leq N$ and $P_{N^*N^*} = 1$. The transition matrix $Q$ for inspection is given by

$$Q_{ij} = P(X_{n+1} = j \mid X_n = i, A_n = I) = \begin{cases} P_{ij}, & j < M \\ 0 & M \leq j \leq N \\ \sum_{k=M}^{N} P_{ik} & j = N^* \end{cases}$$

for $i \leq N$, and $Q_{N^*N^*} = 1$. We further define the inspection cost $I_{N^*}$ and the non-inspection cost $0_{N^*}$ to be zero. Thus, when the process is inspected and discovered to be entering a state $j \geq M$, it is placed in state $N^*$ and no further costs are incurred. The inspection cost $I_j$, $j \geq M$ is considered to include any cost of terminating the process.

In this chapter we will consider the number of potential inspection opportunities $L$ to be infinite. We will discount costs at each transition (inspection or non-inspection) so that $\alpha \leq 1$ and $\beta = 1$ .

In light of the above assumptions, the optimality equation (2.2.2) reduces to

$$(3.1) \quad V^*(i,t) = \min \left\{ \sum P_{ij}^{t+1} O_j + \alpha V^*(i,t+1) ; \sum P_{ij}^{t+1} (I_j + \alpha V^*(j,0)) \right\} .$$

We will show that the optimal policy for both processes is specified by a nonincreasing sequence of critical numbers $t^*(i)$ , such that it is optimal to inspect in state $(i,t)$ if and only if $t \geq t^*(i)$ .

## 3.2 Structural Results for Model I

In order to use the results of Section 2.3, we will again make assumptions A1 and A2 of that section: The transition matrix $P$ is upper triangular with $P_{ii} < 1$ for $i < N$ and $P$ is $TP_2$ .

We will need to show that the structural conditions (SC1) and (SC2) of Section 2.3 hold. We will therefore make the following assumptions on costs:

B1: $I_j \geq O_j$ , $j < M$ , and $I_j - O_j$ is nonincreasing in $j$ , for $j \leq N$ .

B2: $I_j < O_j + \alpha \sum P_{jk} I_k$ for $j \geq M$ .

Assumption B1 means that, for the "good" states $j < M$ , the cost of inspection is larger than the cost of non-inspection, but the the cost of non-inspection increases relative to the cost of inspection as the state becomes higher.

In order to interpret condition B2, we will introduce the auxiliary observable state $(j,-1)$ .

Definition:

We will say that $Y_0 = (j,-1)$ if the value of $X_1$ is known to be $j$ before the decision $A_0$ is chosen.

Note that this is consistent with our definition of $(j,t)$ , $t \geq 0$ .

Assumption B2 is equivalent to $V(j,-1,\pi_0) < V(j,-1,\pi_1)$ or $\Delta(j,-1) > 0$ , for $j \geq M$ . This means that, given the next state will be $j \geq M$ , immediate inspection and stopping is less costly than non-inspection followed by immediate inspection and stopping.

If inspection is strictly optimal for state $(j,-1)$ , $j \geq M$ then B2 must hold. The reserve implication is also true, as a result of the proof of Theorem 2.3.

For the case $M = N$ , B2 reduces to $I_N < \frac{O_N}{1-\alpha}$ , i.e. inspection (and stopping) is cheaper than non-inspection forever.

Finally, if $\alpha = 1$ , $O_j > 0$ , and $I_j$ is nondecreasing in $j$ , $j \geq M$ , then B2 is satisfied.

We are now ready to state the main theorem of this section.

Theorem 3.2:

Under the assumptions A1, A2, B1 and B2: There exist critical numbers $t^*(i) < \infty$ nonincreasing in $i$ , $i < M$ , such that, defining $\pi^*$ to be the policy which inspects at $(i,t)$ if and only if

$t \geq t^*(i)$ , $\pi^*$ is optimal.

Furthermore $\pi^*$ is a one-stage look-ahead policy, i.e. $\pi^*$ inspects at $(i,t)$ if and only if inspection at $(i,t)$ is cheaper than non-inspection, followed by inspection at $(i,t+1)$ , subsequent actions being optimal.

Proof:

By Theorem 2.3, it is sufficient to establish the structural conditions (SC1) and (SC2). We will need the following two lemmas:

Lemma 3.2.1:

Information inequality: $V^*(i,t) \geq \sum P_{ij}^{t-t_0} V^*(j,t_0)$ for all $-1 \leq t_0 \leq t$ .

Proof:

Let $\pi$ be the policy "proceed optimally under the assumption that $Y_0 = (i,t)$ ." Let $C(\pi) = \sum \alpha^n C(Y_n(\pi), A_n(\pi))$ be the (random) total discounted cost under $\pi$ . Thus $V^*(i,t) = V(i,t,\pi) = E(C(\pi) \mid Y_0 = (i,t)) = E(C(\pi) \mid X_{-t} = i) =$ (using the Markovian property of the process) $\sum P_{ij}^{t-t_0} E\left(C(\pi) \mid X_{-t_0} = j\right) = \sum P_{ij}^{t-t_0} V(j,t_0,\pi) \geq \sum P_{ij}^{t-t_0} V^*(j,t_0)$ . ■

Remark:

We call this the information inequality because it states that, if the decision maker is informed of the outcome of the $(t - t_0)$ step transition from state $i$ , and acts optimally on that information, this action cannot increase total expected cost.

Lemma 3.2.2:

(a) For any sequence $\Delta_0, \Delta_1, \ldots, \Delta_N$ and any $i$,

$$\lim_{t\to\infty} \sum P_{ij}^t \Delta_j = \Delta_N$$

(b) $\lim_{t\to\infty} V^*(i,t) = I_N$ .

Proof:

From Assumption A1, the states $\{0,1, \ldots, N-1\}$ are transient and state $N$ is absorbing. It follows that

$$\lim_{t\to\infty} P_{ij}^t = \begin{cases} 0 & j < N \\ 1 & j = N . \end{cases}$$

This proves (a).

To prove (b), let $\varepsilon > 0$ be given and let $T$ be sufficiently large that $t \geq T$ implies $P_{iN}^t \geq 1 - \varepsilon$ . Then for $t \geq T$ , $V^*(i,t) \leq V(i,t,\pi_0) = \sum P_{ij}^{t+1}(I_j + \alpha V^*(j,0)) \leq M\varepsilon + (1-\varepsilon)I_N$ where $M = \max \left\{I_j + \alpha V^*(j,0)\right\}$ is a constant independent of $\varepsilon$ . ($V^*(N,0) = 0$ since $(N,0)$ is identified with $(N^*, 0)$) . It follows that

$$\limsup_{t\to\infty} V^*(i,t) \leq I_N .$$

To obtain the reverse inequality, we will use Lemma 3.2.1 to obtain $V^*(i,t) \geq \sum P_{ij}^{t+1} V^*(j,-1)$ . Hence if $t \geq T$ , we have $V^*(i,t) \geq (1-\varepsilon)V^*(N,-1)$ . Since $N$ is an absorbing state, either it is optimal not to inspect at all future times, at cost

$\frac{O_N}{1-\alpha}$ , or it is optimal to inspect (and stop) at cost $I_N$ . By B2, $I_N < \frac{O_N}{1-\alpha}$ , so inspection is optimal and $V^*(N,-1) = I_N$ . Thus $T \geq t$ implies $V^*(i,t) \geq (1-\varepsilon)I_N$ . It follows that $\liminf_{t \to \infty} V^*(i,t) \geq I_N$ , and the result is proved.■

Proof of Theorem 3.2 (continued):

To establish (SC1) we must show that $\Delta(j,0)$ has the SCP in $j$ . As in the proof of Theorem 2.3, we may express $\Delta(j,0) = \sum P_{jk}\Delta(k,-1)$ . By Lemma 2.3, if $\Delta(k,-1)$ satisfies the SCP, so does $\Delta(j,0)$ .

Notice that $\Delta(k,-1) = V(k,-1,\pi_1) - V(k,-1,\pi_0) = O_k + \alpha \sum P_{k\ell}(I_\ell + \alpha V^*(\ell,0)) - (I_k + \alpha V^*(k,0))$ . If $k \geq M$ , then $V^*(\ell,0) = 0$ for $\ell \geq k$ , so that $\Delta(k,-1) = O_k + \alpha \sum P_{k\ell}I_\ell - I_k > 0$ by Assumption B2.

Thus, to establish that $\Delta(k,-1)$ has the SCP, it suffices to show that $\Delta(k,-1) \leq 0$ for $k < M$ . We consider two cases.

Case I:

$V^*(k,0) = \sum P_{k\ell}(I_\ell + \alpha V^*(\ell,0))$ . (Inspection is optimal in observed state $(k,0)$) . In this case $\Delta(k,-1)$ reduces to $O_k - I_k \leq 0$ (by Assumption B1).

Case II:

$V^*(k,0) = \sum P_{k\ell})_\ell + \alpha V^*(k,1)$ . (Non-inspection is optimal in state $(k,0)$) . In this case $\Delta(k,-1)$ becomes

$$\left[\alpha \sum P_{k\ell}(I_\ell - O_\ell) - (I_k - O_k)\right] + \alpha^2 \left[\sum P_{k\ell} V^*(\ell,0) - V^*(k,1)\right] .$$

The first term is non-positive from B1 because $(I_j - O_j)$ is non-increasing in $j$ and $I_k - O_k \geq 0$. The second term is non-positive as a result of Lemma 3.2.1. Thus $\Delta(k,-1) \leq 0$ for $k < M$. The first structural condition (SC1) is now established.

Remark:

Assumption B1 can be weakened to

B1': $\alpha \sum P_{k\ell}(I_\ell - O_\ell) \leq I_k - O_k \geq 0$ , $k < M$ .

To establish the second structural condition (SC2), we consider the optimality equation (3.1). Let $\Delta_{it}$ be the difference of the terms corresponding to non-inspection and inspection, i.e.

$$\Delta_{it} = \sum P_{ij}^{t+1} O_j + \alpha V^*(i,t+1)$$
$$- \sum P_{ij}^{t+1}(I_j + \alpha V^*(j,0)) .$$

We must show that for any $i$, for sufficiently large $t$, inspection is strictly optimal when in state $(i,t)$. By Theorem 2.2, it will be sufficient to show that $\lim_{t\to\infty} \Delta_{it} > 0$. Rewriting $\Delta_{it}$ as

$$\Delta_{it} = \sum P_{ij}^{t+1}(O_j - I_j) + \alpha\left(V^*(i,t+1) - \sum P_{ij}^{t+1} V(j,0)\right)$$

we apply Lemma 3.2.2 and obtain

$$\lim_{t\to\infty} \Delta_{it} = (O_N - I_N) + \alpha(I_N - V^*(N,0)) .$$

Since $V^*(N,0) = 0$ , this becomes

$$\lim_{t\to\infty} \Delta_{it} = O_N + \alpha I_N - I_N > 0 \quad \text{by B2.}$$

Hence (SC2) is established. By Theorem 2.3, the proof is now complete. ■

## 3.3 Structural Results for Model II:

Model II (inspection with repair) can be viewed as a special case of model I by considering the optimal cost after repair, $V^*(0,0)$ , to be a terminal cost, and including it in the inspection-repair cost $I_j$ , $j \geq M$ . In order for $V^*(0,0)$ to be finite, of course in this case we must have $\alpha < 1$ .

We prefer, for reasons of clarity to let $I_j$ , $j \geq M$ include only the cost of inspection and repair, and leave $V^*(0,0)$ separate. This will require only minor modifications to the proof of Theorem 3.2. We need only modify Assumption B2:

$$\text{B2}': \quad I_j + \alpha V^*(0,0) < O_j + \alpha \sum P_{jk}(I_k + \alpha V^*(0,0)) \; .$$

**Theorem 3.3:**

Under Assumptions A1, A2, B1, and B2', the conclusion of Theorem 3.2 hold.

The condition B2' means that, in state $(j,-1)$ , $j \geq M$ , immediate inspection (and repair) is cheaper than non-inspection followed by inspection, subsequent actions being optimal. From the proof of Theorem 2.3, condition B2' is equivalent to inspection being optimal for all $(j,t)$ , $j \geq M$ , $t \geq -1$ .

The problem with B2' is that it involves $V^*(0,0)$, which is not readily obtainable from the parameters of the process. However, if an upper bound $\bar{V} \geq V^*(0,0)$ is available, a sufficient condition for B2' is that the inequality hold, with $\bar{V}$ substituted for $V^*(0,0)$ :

$$\text{B2}'': \quad I_j + \alpha\bar{V} < O_j + \alpha \sum P_{jk}(I_k + \alpha\bar{V}) .$$

Lemma 3.3:

If B2'' holds with $\bar{V} \geq V^*(0,0)$, then B2' holds.

Proof:

Immediate, since the coefficient of $\bar{V}$ is $\alpha$ on the L.H.S., and $\alpha^2$ on the R.H.S.■

Remark:

One such upper bound $\bar{V}$ can be obtained by considering the policy $\pi$ : "never inspect." The expected cost of acting according to this policy starting in state $(0,0)$ is $V(0,0,\pi) = \sum_{t=0}^{\infty} \alpha^t \sum_j P_{0j}^{t+1} O_j$, which is of course not smaller than $V^*(0,0)$.

# CHAPTER 4

# AN INVENTORY MODEL WITH RANDOM PRODUCTION

## 4.1 Introduction

We consider the usual dynamic inventory problem of minimizing production, shortage, and holding costs over time, subject to random, periodic demands. (See for example [ 5 ]). We impose the added feature, however, that when producing (or ordering) the amount actually added to stock is random.

The simplest case of this model is the binomial case. Suppose each item produced has a fixed probability $q$ of being unusable, independently of all other items produced. It follows that if production level $n$ is chosen the number of usable items produced will be distributed Binomial $(n,p)$ , where $p = 1 - q$ .

More generally, we will also consider a model in which production occurs in discrete batches of random size $p$ . Batch sizes are assumed independent with a common distribution $P(p = j) = p_j$ . When production level $n$ is chosen, the number of usable items produced is $\sigma(n) = \sum_1^n p_j$ . The binomial model is, of course, a special case of this with $p_0 = q$ and $p_1 = p$ . Another special case is the Poisson model, in which $\sigma(n)$ is distributed Poisson $(n\lambda)$ . This corresponds to $p_j = \frac{\lambda^j e^{-\lambda}}{j!}$ , $j = 0,1, \ldots$ .

We will assume that demands occuring in each period are independent, discrete random variables $\xi_i \geq 0$ , with a common density $P(\xi_i = j) = q_j$ . Unsatisfied demands will be assumed backlogged (carried over to the next period). Thus, inventories in successive periods are linked by the equations

(4.1.1) $$x_{i+1} = y_i - \xi_i = x_i + \sigma_i - \xi_i \; .$$

Here $x_i$ is the initial inventory in period $i$ and $y_i = x_i + \sigma_i$ is the inventory after production but before demand is satisfied. A negative value of $x_i$ signifies a cumulative excess of demand over production. We are implicitly assuming that there is no lag in production, i.e. production undertaken during the current period will be available to meet that period's demand.

The cost of producing at level $n$ , $c(n)$ , is assumed to be linear with no set-up cost, i.e. $c(n) = nc$ , $c > 0$ .

A holding cost $h(y_i - \xi_i)$ is incurred whenever the inventory after production exceeds demand. A shortage cost $k(\xi_i - y_i)$ is incurred whenever there is insufficient inventory to meet demand. We will assume that $h(\cdot)$ and $k(\cdot)$ are nondecreasing functions which vanish for negative arguments.

If initial inventory is $x_i$ and production level $n_i$ is chosen, the cost for period $i$ will be

$$g(x_i,n_i,\sigma_i,\xi_i) = cn_i + h(x_i + \sigma_i - \xi_i) + k(\xi_i - x_i - \sigma_i) \; .$$

We seek a policy $\pi$ (a rule for choosing $n_i$) to minimize the expected total discounted cost $E_\pi \left[ \sum_{\ell=0}^{L-1} \beta^\ell g(x_\ell,n_\ell,\sigma_\ell,\xi_\ell) \mid x_0 \right]$ . The number of periods $L$ may be either finite or infinite. The expectation is with respect to the distribution of $\xi_i$ and the distribution of $\sigma_i$ induced by $\pi$ through choice of $n_i$ .

## 4.2 Formulation of the Inventory Model as a Markov Inspection Process

We will now show that the inventory process with random production described above, fits the Markov inspection model of Chapter 2.

The underlying state of the process is the current inventory $x$. After observing $x$, we must decide how many batches to produce. We may consider the decision as being made sequentially as follows: Suppose $t$ batches have been produced, but the number of usable items they contain, $\sum_{j=1}^{t} \rho_j$, has not yet been determined. This corresponds to the observable state $(x,t)$. We may either produce an additional batch, or terminate production. The former action corresponds to non-inspection, at cost $C$, and yields the new observable state $(x,t+1)$. The latter action corresponds to inspection, since the new state $(y,0)$, where $y = x + \sum_{i=1}^{t} \rho_i - \xi$ becomes known.

The transition matrix associated with production (non-inspection) is given by

$$P_{xx'} = \begin{cases} p_{x'-x} & x \leq x' \\ 0 & x > x' . \end{cases}$$

The transition matrix associated with termination and satisfaction of demand (inspection) is given by

$$Q_{xx'} = \begin{cases} q_{x-x'} & x \geq x' \\ 0 & x < x' . \end{cases}$$

The costs of non-inspection and inspection, respectively are $O_x = C$ and $I_x = h(x) + k(-x)$.

With these identifications our inventory process becomes a Markov Inspection Process as defined in 2.1.

Remark:

The model of Chapter 2 permits greater generality, in that the size of the j-th batch produced might be allowed to depend on previous production $x_i + \sum_{k=1}^{j-1} p_j$. This would give rise to a more general transition matrix than $P_{xx'}$ given above. However, as it is difficult to see any application for this added generality, we will not pursue it further.

We seek to minimize the expected total discounted cost over $L$ periods, discounting at a rate of $\beta$ per period. The optimality equation (2.2.2) specializes to

$$\begin{aligned} V_L(x,t) = \min \Big\{ & C + V^*(x,t+1,L); \\ & \sum P^t_{xy} Q_{yz} \left[h(z) + k(-z) + \beta V^*(z,0,L-1)\right] \Big\} . \end{aligned} \tag{4.2.1}$$

Having identified our inventory model as a Markov Inspection Process, we wish to apply the results of Chapter 2 to find conditions for the optimality of a monotone policy. In other words, when is the optimal ordering level a decreasing function of current inventory, as intuition suggests?

Note that the matrix $P$ is upper triangular. If the distribution $\{p_j\}$ is such that $P$ is $TP_2$ then Assumptions A1 and A2 of Chapter 2 are satisfied. (Such distributions are called Polya frequency functions of order 2). $(PF_2)$. See [ 6 ].

Furthermore, since the unit production cost is $c > 0$, the second structural condition (SC2) must hold. Otherwise, for some observed state $(x,t)$, continual production is optimal. But this policy has infinite expected cost, which produces a contradiction.

Thus the general structure theorem of Chapter 2 (Theorem 2.3) applies, provided we assume the first structural condition (SC1).

Theorem 4.2:

For the $L$ period production problem with $PF_2$ production distribution $\{p_j\}$, linear ordering cost, and general holding and penalty costs, if there is a critical number $x_0$ such that ordering zero is preferred to ordering one when $x \geq x_0$ and not otherwise, (subsequent orders being optimal), then the optimal order size $n^*(x,L)$ is nonincreasing in initial inventory $x$.

Furthermore, the cost of the policy $\pi_n$ : (order $n$, then proceed optimally), $V(x,0,L,\pi_n)$, is quasi-convex in $n$. (A function $f(n)$ is quasi-convex if $f(n+1) \leq f(n)$ for $n <$ some $n^*$ and $f(n+1) \geq f(n)$ for $n \geq n^*$).

The fact that $V(x,0,L,\pi_n)$ is quasi-convex in $n$ is an immediate consequence of the optimality of one step look-ahead policies (see (2.3).■

In the remainder of this chapter we will let $V(x,0,L,\pi_n) = V(x,L,\pi_n)$ and $V^*(x,0,L) = V^*(x,L)$. Also we will deal only with the binomial case henceforth.

## 4.3 The Binomial, Single Period Case

We first note that when production is Binomial the associated transition matrix has the form

$$P_{xx'} = \begin{cases} q & x' = x \\ p & x' = x+1 \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $P$ satisfies hypotheses A1 and A2 of 2.3; $P$ is upper triangular and totally positive of order 2.

In order to apply Theorem 4.2, it is sufficient to verify the structural condition (SC1). That is, we must show that there is a critical number $x_1$ such that $V(x,\pi_1) \leq V(x,\pi_0)$ for $x < x_i$ and $V(x,\pi_1) > V(x,\pi_0)$ for $x \geq x_1$. (We suppress $L$ for the single period case). This will imply that the optimal order size $n^*(x) = 0$ for $x \geq x_1$ and $n^*(x) > 0$ for $x < x_1$.

To show that the first structural condition holds, it will be necessary to make the following assumptions:

A3: The demand distribution $\{q_j\}$ is a Polya frequency function of order 2 $(PF_2)$. That is, the matrix $Q_{ij} = q_{i-j}$ is $TP_2$ (see Assumption A2 of 2.3).

A4: The function $\ell(x) = \frac{c}{p} x + h(x) + k(-x)$ is quasi-convex with a minimum at $x = 0$, and strictly increasing for $x \geq 0$. (This is equivalent to $\ell(x+1) - \ell(x)$ satisfying the SCP in $x$).

Assumption A3 immediately yields the analog of Lemma 2.3:

Lemma 4.3.1:

If $\{q_j\}$ is a $PF_2$ distribution and $m_j$ satisfies the SCP in $j$, then $\sum q_j m_{i-j}$ satisfies the SCP in $i$.

Proof:

Apply Lemma 2.3 to the matrix $Q_{ij}$ and make a change of variables.■

Assumption A4 has the following interpretation: Consider the certainty equivalent problem where we pay $\frac{c}{p}$ per item and all orders are filled exactly. Suppose demand in a period is known. The function $\ell(x)$ represents the relative cost of producing an excess $x$ over demand. Assumption A4 implies that $\ell(x)$ is nonincreasing in $x$ for $x \leq 0$, which means that each additional unit cost of production $\frac{c}{p}$ is more than compensated for by a decrease in penalty (shortage) cost $k(-x)$. Thus $\frac{c}{p} x + k(-x) \downarrow x$, $x \leq 0$. Clearly, when $x \geq 0$, $\frac{c}{p} x + h(x) \uparrow x$ (strictly, for $c > 0$).

Theorem 4.3:

For the one period problem with binomial production, Assumption A3 and A4 imply that the optimal production level $n^*(x)$ is a non-increasing function of initial inventory $x$. Furthermore, for each $x$, the cost of the policy "produce $n$," $V(x,\pi_n)$, is quasi-convex in $n$, i.e. $V(x,\pi_{n+1}) \leq V(x,\pi_n)$ for $n < n^*(x)$ and $V(x,\pi_{n+1}) \geq V(x,\pi_n)$ for $n \geq n^*(x)$.

Proof:

By Theorem 4.2, we need only show that the first structural condition (SC1) is satisfied, (SC2), (A1), and (A2) having already been shown. Equivalently, we must show that for some $x_1$, $V(x,\pi_1) \leq V(x,\pi_0)$ when $x < x_1$ and not when $x \geq x_1$. We must show that $\Delta(x) = V(x,\pi_1) - V(x,\pi_0)$ satisfies the SCP in $x$. By Assumption A4, for $\ell(x) = \frac{c}{p} x + h(x) + k(-x)$, the successive differences $\ell(x + 1) - \ell(x)$ satisfy the SCP in $x$. By Assumption A3 and Lemma 4.3, so does

$$\sum q_z [\ell(x + 1 - z) - \ell(x - z)] =$$

$$E \{\ell(x + 1 - \xi) - \ell(x - \xi)\} =$$

$$\frac{c}{p} + \ell_1(x + 1) - \ell_1(x) \text{ , where } \ell_1(x) = \tag{4.3.1}$$

$$E \{h(x - \xi) + k(\xi - x)\}$$

is the expected storage plus shortage cost when inventory is $x$ after production. But

$$\Delta(x) = V(x,\pi_1) - V(x,\pi_0) =$$

$$[c + q\ell_1(x) + p\ell_1(x + 1)] - \ell_1(x) =$$

$$c + p(\ell_1(x + 1) - \ell_1(x)) .$$

Since the expression in (4.3.1) is just $\frac{1}{p} \Delta(x)$, we conclude that $\Delta(x)$ also satisfies the SCP, and the proof is concluded.■

Remark:

Note that no assumptions on the form of $h(x)$ and $k(x)$ are made, other than A4.

There is an alternate approach to obtaining the optimality of a monotone policy, assuming convexity of $h(x)$ and $k(x)$, but without restriction on the demand distribution. This is the approach followed in Karlin, Arrow and Scarf [ 4 ]. This method does not extend to the multi-period case, however, because $V^*(x)$ need not be convex when $h(x)$ and $k(x)$ are.

Example 4.3.1: (Non-Convexity of the Optimal Cost Function $V^*(x)$)

Let $V^*(x) = V^*(x,1)$ be the optimal cost function for the one period problem with initial inventory $x$. Let $V(x,\pi_n)$ be the expected cost under the policy "produce $n$ items."

Suppose $p = \frac{1}{2}$, $c = 1$, $h(u) \equiv 0$ and $k(u) = \begin{cases} 3u & u \geq 0 \\ 0 & u \leq 0 \end{cases}$.
Assume also that the demand $\xi \equiv 2$.

Clearly $V^*(2) = 0$ and $n_1^*(2) = 0$.

For $x = 1$ we compute as follows:

$$V(x,\pi_0) = k(1) = 3$$

$$V(x,\pi_1) = c + qk(1) = 2\ 1/2$$

$$V(x,\pi_2) = 2c + q^2(k(1)) = 2\ 3/4$$

By quasi-convexity in $k$ of $V(x,\pi_k)$ (Theorem 4.3), we may conclude that $V^*(1) = 2\ 1/2$ and $n^*(1) = 1$.

Similarly, for $x = 0$ we compute

$$V(0,\pi_0) = k(2) = 6$$

$$V(0,\pi_1) = c + qk(2) + pk(1) = 5\ 1/2$$

$$V(0,\pi_2) = 2c + q^2k(2) + 2pqk(1) = 5$$

$$V(0,\pi_3) = 3c + q^3k(2) + 3q^2pk(1) = 4\ 7/8$$

$$V(0,\pi_4) = 4c + q^4k(2) + 4q^3pk(1) = 5\ 1/8$$

Again by Theorem 4.3, since $V(0,\pi_3) < V(0,\pi_4)$ and $V(0,\pi_3) < V(0,\pi_j)$, $j < 3$, we can conclude that $V^*(0) = 4\ 7/8$ and $n^*(0) = 3$.

Thus $V^*(x)$ can be tabulated:

| x | $V^*(x)$ | $n^*(x)$ |
|---|---|---|
| 0 | 4.875 | 3 |
| 1 | 2.5 | 1 |
| 2 | 0 | 0 |

We see immediately that $V^*(x)$ is strictly concave on the set $\{0,1,2\}$.

When the holding and shortage costs are linear, say $h(u) = hu$ and $k(u) = ku$, respectively, we can determine $n^*(x)$ by the following reasoning: Suppose we have produced, but not inspected, $n$ items. Let $\sigma(n)$ be the Binomial $(n;p)$ random variable denoting usable output. The expected cost of producing an additional item is $c + hpP(x + \sigma(n) \geq \xi)$, which is nondecreasing in $n$. The expected gain from producing an additional item is $kpP(x + \sigma(n) < \xi)$ which is nonincreasing in $n$. It follows that

$$n^*(x) = \min \{n : kpP(x + \sigma(n) < \xi) \leq c + hpP(x + \sigma(n) \geq \xi)\}$$

$$= \min \left\{n : P(\sigma(n) < \xi - x) \leq \frac{c + ph}{p(k + h)}\right\} .$$

From this relationship several interesting conclusions can be drawn.

Lemma 4.3.2:

In the case where all costs are linear:

(a) $n^*(x)$ depends on $c$, $h$, and $k$ only through $\frac{c + ph}{p(k + h)}$.

(b) $n^*(x)$ is nonincreasing in $c$ and $h$ and nondecreasing in $k$.

(c) Production is optimal only when $p \geq p_0 = \frac{c}{k}$.

The dependence of $n^*(x)$ on $p$ is more subtle. The next example (4.3.2) shows that $n^*(x)$ need not be a monotone function of $p$ for a given $x$, even when $\xi \equiv D$ (deterministic demand).

Note that it is optimal to produce if and only if $p \geq p_0 = \frac{c}{k}$, provided $x < D$. The optimal production level $n^*$ appears to be unimodal as a function of $p$ for fixed $x$. Thus, when $p$ is near $p_0$, and when $p$ is near $1$, $n^* = D - x$, but $n^*$ may be strictly larger than $D - x$ for intermediate values of $p$. For $p$ near $p_0$, it is optimal to produce $D - x$, but no "spares." This is because $p$ is so small that the expected reduction in shortage cost from producing a spare is less than the unit production cost $c$.

In Example 4.3.2 the optimal cost $V^* = V^*(0)$ is a nonincreasing function of $p$. This is true in general for the one period case.

Lemma 4.3.3:

For fixed inventory $x$, the optimal cost $V^*(x)$ is a non-increasing function of $p$.

Proof:

When the holding cost is zero, for a fixed ordering level $n$ the expected cost is $cn + kE \max(\xi - x - \sigma(n), 0)$. This is a continuous, nonincreasing function of $p$. The result follows immediately in this case, since $V^*(x)$ is the minimum of the above functions.

When the holding cost $h$ is non-zero the expected cost of an order size $n$ is $cn + kE \max(\xi - x - \sigma(n), 0) + hE \max(x + \sigma(n) - \xi, 0)$ which increases for $p > p_n$ and decreases for $p < p_n$ for some $p_n$. It can be shown that $n^* < n$ for $p > p_n$, which would imply the result. We omit the proof.

Example 4.3.2: Variation of Optimal Order Size $n^*(x)$ and Optimal Cost $V^*(x)$ as a Function of $p$ for Fixed $x$

Consider the one period linear problem with $c = 1$ , $k = 4$ , $h = 0$ , $x = 0$ , and $D = 2$ .

Note that for $p \leq \frac{c}{k} = .25$ , $n^* = 0$ and $V^* = 2k = 8$ . For $p = 1$ , $n^* = 2$ and $V^* = 2c = 2$ . For intermediate values of $p$ , $n^*$ and $V^*$ are given in the following table. Note that $n^*$ rises to a maximum and then decreases, while $V^*$ is continuously decreasing in the interval $[0,1]$ .

## 4.4 The Multiperiod, Binomial Case: A Counter-Example and a Result

The result of the previous section concerning the optimality of monotone ordering policies for the single period case, does not extend without restriction to the multiperiod case. The following example is a 2 period problem with linear ordering, holding, and shortage costs and a constant demand, for which a monotone policy is not optimal.

Example 4.4: An Example in Which a Monotone Policy is Not Optimal

We set $c = 1$ , $k(u) = 3u$ , $h(u) = hu$ , $p = \frac{1}{2}$ , and $\xi \equiv 2$ . Assume $h \leq 1$ .

Consider first the single period case. Note that with $h = 0$ this is identical to Example 4.3.1. A similar calculation shows the following result:

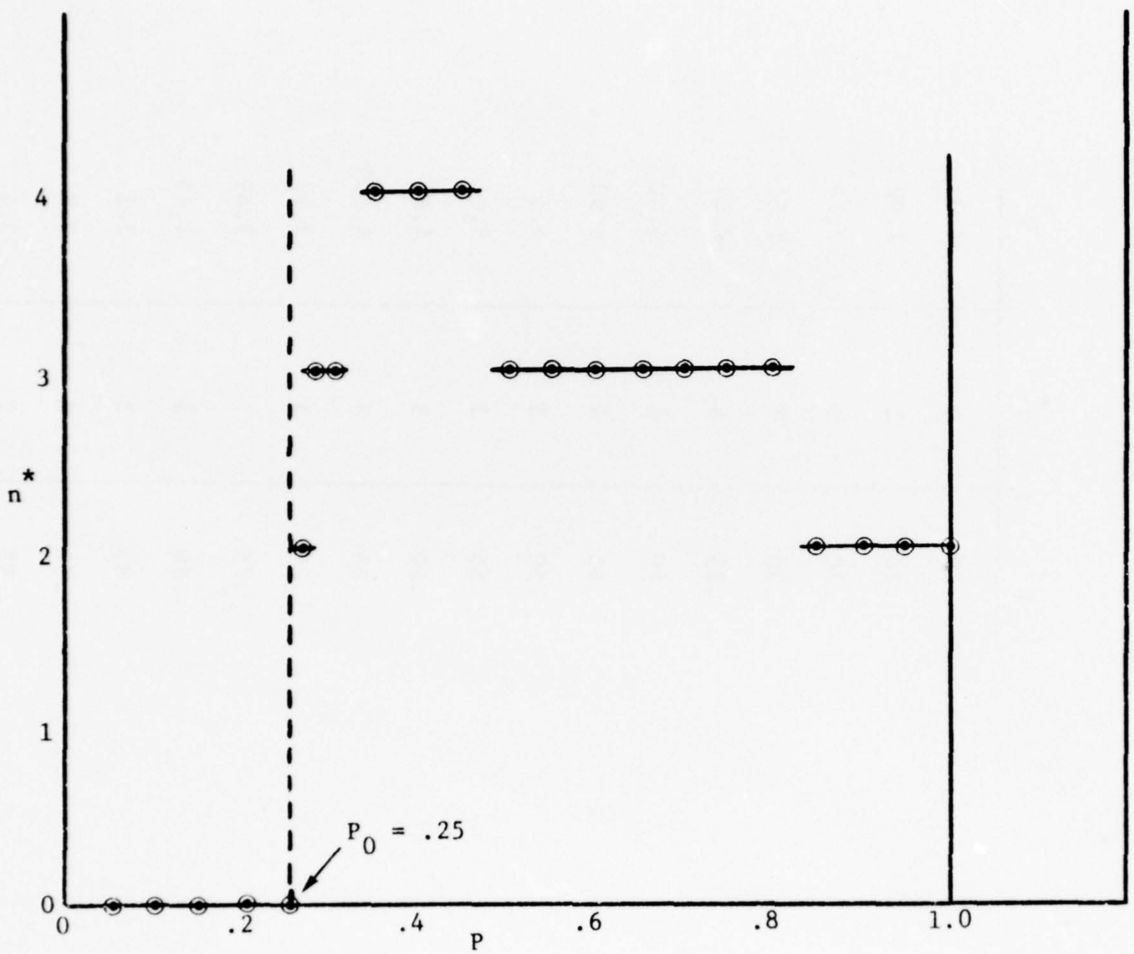
EXAMPLE 4.3.2

TABLE OF $n^*$ AND $V^*$ AS A FUNCTION OF P

| P | $n^*$ | $V^*$ |
|---|---|---|
| 0-.25 | 0 | 8.0 |
| .26 | 2 | 7.92 |
| .28 | 3 | 7.73 |
| .30 | 3 | 7.51 |
| .35 | 4 | 6.97 |
| .40 | 4 | 6.42 |
| .45 | 4 | 5.93 |
| .50 | 3 | 5.5 |
| .55 | 3 | 5.07 |
| .60 | 3 | 4.66 |
| .65 | 3 | 4.30 |
| .70 | 3 | 3.97 |
| .75 | 3 | 3.69 |
| .80 | 3 | 3.45 |
| .85 | 2 | 3.2 |
| .90 | 2 | 2.8 |
| .95 | 2 | 2.4 |
| 1.0 | 2 | 2.0 |

EXAMPLE 4.3.2

PLOT OF $n^*$ AS A FUNCTION OF P



| $x$ | $n^*(x,1)$ | $V^*(x,1)$ |
|---|---|---|
| 2 | 0 | 0 |
| 1 | 1 | 2.5 |
| 0 | 3 | $4.875 + .125h$ |

Analysis of the two period case is dependent on the value of $h$ :

| $x$ | $n^*(x,2)$ | $V^*(x,2)$ | |
|---|---|---|---|
| 4 | 0 | $2h$ | |
| 3 | 1 | $2.25 + 1.5h$ | $h \leq 1/2$ |
| | 0 | $2.5 + h$ | $h \geq 1/2$ |
| 2 | 2 | $4.46875 + 1.03125h$ | $h \leq .448$ |
| | 0 | $4.875 + .125h$ | $h \geq .448$ |

Thus we see that for $.448 \leq h \leq .5$ , $n^*(2,2) = 0$ and $n_2^*(3,2) = 1$ . The optimal policy is not monotone.

A further analysis of the two period case reveals that a one-stage look-ahead policy is not optimal. Let $\pi_k$ be the policy: order $k$ during the first period, then continue optimally. For this example the following behavior may be observed, given initial inventory of 2 with two periods remaining.

$$\pi_2 < \pi_1 < \pi_0 \qquad 0 \leq h \leq .429$$

$$\pi_2 < \pi_0 < \pi_1 \qquad .429 \leq h \leq .448$$

$$\pi_0 < \pi_2 < \pi_1 \qquad .448 \leq h \leq .467$$

$$\pi_0 < \pi_1 < \pi_2 \qquad .467 \leq h$$

(Here $\pi_i < \pi_j$ means $V(2,2,\pi_i) < V(2,2,\pi_j)$). For $.429 \leq h \leq .467$, $V(2,2,\pi_n)$ is not quasi-convex in $n$ and the one stage look-ahead policy is not optimal.

Example 4.4b: A Second Example in Which a Monotone Policy is Not Optimal

We modify the previous example by setting the holding cost $h = 0$ and imposing a discount factor $\beta \leq 1$.

We consider the two period case.

| $x$ | $n^*(x,2)$ | $V^*(x,2)$ | |
|---|---|---|---|
| 4 | 0 | 0 | |
| 3 | 0 | $2.5\beta$ | $\beta \leq .8$ |
| | 1 | $1 + 1.25\beta$ | $\beta \geq .8$ |
| 2 | 0 | $4.875\beta$ | $\beta \leq .831$ |
| | 2 | $2 + 2.46875\beta$ | $\beta > .831$ |

Thus we see that in the interval $\beta = .8$ to $\beta = .831$, a monotone policy is not optimal.

The above example illustrates the effect of the concavity of $V^*(x,1)$ on optimal policies for the two period case. The closer initial inventory is to total demand $2D$, the greater the expected reward for ordering an additional item. This leads to the possible optimality of non-monotone policies in the case of positive holding cost or discount factor $\beta < 1$.

However, in the non-discounted multiperiod case with no holding cost, a monotone policy is optimal.

Before proving this result we state a lemma which is valid under very general conditions:

Lemma 4.4:

If $n^*(x,\ell) > 0$ , then

$$V^*(x,\ell) \geq \frac{c}{p} + V^*(x + 1,\ell) \ .$$

Proof:

Given the option to inspect the first item produced and revise initial lot size accordingly, the optimal cost for $\ell$ periods, starting with inventory $x$ would be

$$c + qV^*(x,\ell) + pV^*(x + 1,\ell) \leq V^*(x,\ell) \ .$$

The R.H.S. represents the optimal cost in the absence of such an option.

Remark:

This result shows that, when it is optimal to order, the value of an additional item in stock is at least as large as its "certainty equivalent price," $c/p$ .

Theorem 4.4:

For the $\ell$ period, undiscounted problem with binomial production, if there is no holding cost $(h(u) \equiv 0)$ , then assumptions A3 and A4 imply:

(a) The optimal initial production lot size $n^*(x,\ell)$ is a nonincreasing function of initial inventory $x$.

(b) The expected cost of the policy $\pi_n$ : produce $n$, then choose all subsequent lot sizes optimally, $V(x,\ell,\pi_n)$, is quasi-convex in $n$.

Proof:

By Theorem 4.2, we need only show that $\Delta(x,\ell)$ satisfies the SCP in $x$. Equivalently we must show the existence of a critical number $x_\ell$ such that $V(x,\ell,\pi_1) \leq V(x,\ell,\pi_0)$ for $x < x_\ell$ and $V(x,\ell,\pi_1) > V(x,\ell,\pi_0)$ for $x \geq x_\ell$.

Proceeding inductively on $\ell$, we note that the result has previously been shown for $\ell = 1$. Assume that $\Delta(x,\ell - 1)$ satisfies the SCP in $x$ and $x_{\ell-1} = \min\{x : \Delta(x,\ell - 1) > 0\} \geq 0$.

By the induction hypothesis, $n^*(x,\ell - 1) = 0$ for $x \geq x_{\ell-1}$. Thus $V^*(x,\ell - 1)$ is also the expected $\ell - 1$ period cost when inventory is $x$ after initial production. For $x \geq x_{\ell-1}$, $V(x,\ell - 1,\pi_0) < V(x,\ell - 1,\pi_1)$, or equivalently

$$V^*(x,\ell - 1) < c + pV^*(x + 1,\ell - 1) + qV^*(x,\ell - 1) .$$

This shows that $\frac{c}{p}x + V^*(x,\ell - 1)$ is strictly increasing in $x$ for $x \geq x_{\ell-1}$.

Analogously to the proof of Theorem 4.3, let $m(x) = \frac{c}{p}x + k(-x) + V^*(x,\ell - 1)$. Since $k(-x) = 0$ for $x \geq x_{\ell-1}$ $(\geq 0)$, the preceding remark implies that $m(x)$ is strictly increasing for $x \geq x_\ell$.

By Lemma 4.4 and monotonicity of $k(-x)$, $m(x)$ is nonincreasing for $x \leq x_{\ell-1}$. We have shown that $m(x+1) - m(x)$ satisfies the SCP in $x$. As in the proof of Theorem 4.3, this implies that $\Delta(x,\ell) = pE\{m(x+1-\xi) - m(x-\xi)\}$ satisfies the SCP in $x$.

To complete the induction, it will suffice to prove that $x_\ell = \min\{x : \Delta(x,\ell) > 0\} \geq x_{\ell-1}$. We have shown that $m(x+1) - m(x) \leq 0$ for $x \leq x_{\ell-1}$. Thus $\Delta(x,\ell) = pE\{m(x+1-\xi) - m(x-\xi)\} \leq 0$ for $x \leq x_{\ell-1}$. It follows that $x_\ell \geq x_{\ell-1}$, which completes the proof. ■

We have also proven the following:

<u>Lemma 4.4.2</u>:

Under the hypotheses of Theorem 4.4, if it is optimal to order when stock is $x$ and $\ell$ periods remain, it is also optimal to order when $\ell'$ period remain, $\ell' > \ell$.

<u>Remark</u>:

We have not been able to show that the optimal production level $n_\ell^*(x)$ is nondecreasing in $\ell$. This seems to be an open question.

# CHAPTER 5

## SUMMARY AND CONCLUSIONS

We present a general model of a process which requires costly inspection in order to determine the state. At any time two alternatives are available: to inspect the process (which may alter it), or not to inspect.

The optimal time interval between successive inspections is shown to be a decreasing function of the state at last inspection, under conditions specified herein.

Our model is applicable to a wide variety of deteriorating processes, including repairable and non-repairable machines, and a class of inventory problems with uncertain ordering.

The results presented illuminate the structure of optimal inspection policies and are useful in computing such policies.

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